A Early 19th-Century Geometric Construction of a Counterexample to the Jacobian Conjecture

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1 Set-up

Assume that the ground field has characteristic different from 2 and 3. Fix two points

$$0, \infty \in \mathbf{P}^1,$$

and use 0 as the origin on the affine line

$$\mathbb{A} := \mathbf{P}^1 \setminus \{\infty\}.$$

We write points of symmetric powers additively. Thus $y + z$ denotes a point of $\mathrm{Sym}^2(\mathbf{P}^1)$, and $x + y + z$ denotes a point of $\mathrm{Sym}^3(\mathbf{P}^1)$.

Set

$$X = \mathbf{P}^1 \times \mathrm{Sym}^2(\mathbf{P}^1) \simeq \mathbf{P}^1 \times \mathbf{P}^2,$$

and consider the finite morphism

$$f : X \longrightarrow \mathrm{Sym}^3(\mathbf{P}^1) \simeq \mathbf{P}^3, \qquad f(x, Q) = x + Q.$$

Also let

$$g : \mathbf{P}^1 \longrightarrow \mathrm{Sym}^2(\mathbf{P}^1), \qquad g(x) = 2x.$$

Its image

$$\Gamma := g(\mathbf{P}^1) \subset \mathbf{P}^2$$

is the Veronese conic. The composition

$$\mathbf{P}^1 \xrightarrow{(\mathrm{id}, g)} X \xrightarrow{f} \mathrm{Sym}^3(\mathbf{P}^1)$$

is the twisted cubic

$$x \longmapsto 3x.$$

The ramification divisor of f is

$$D = \{(x, Q) : Q \text{ contains } x\}.$$

For each $x \in \mathbf{P}^1$, write

$$D_x := D \cap (\{x\} \times \mathrm{Sym}^2(\mathbf{P}^1)).$$

Let $H \subset \mathrm{Sym}^3(\mathbf{P}^1)$ be the hyperplane whose intersection divisor with the twisted cubic is

$$H \cdot \{3x : x \in \mathbf{P}^1\} = 2[\infty] + [0].$$

Set

$$E := f^{-1}(H), \qquad U := X \setminus (D \cup E).$$

For each $x \in \mathbf{P}^1$, write

$$E_x := E \cap (\{x\} \times \mathrm{Sym}^2(\mathbf{P}^1)).$$

2 The hyperplane as a zero-sum condition

On the affine line $\mathbb{A}$, the hyperplane H has the following geometric interpretation:

$$\boxed{x + y + z \in H \quad \Longleftrightarrow \quad x + y + z = 0,}$$

where the sum on the right is the affine-vector-space sum on $\mathbb{A}$. Indeed, on the diagonal $x = y = z = t$, this condition becomes $3t = 0$. Thus the finite intersection with the twisted cubic occurs at $t = 0$, and the projective closure contributes the remaining intersection multiplicity two at ∞.

Consequently, for finite $x \in \mathbb{A}$ and $Q = y + z$, one has

$$Q \in E_x \quad \Longleftrightarrow \quad x + y + z = 0.$$

3 The first family of lines: tangents to the Veronese conic

For fixed $x \in \mathbf{P}^1$,

$$D_x = \{x + y : y \in \mathbf{P}^1\}.$$

This is a line in $\mathrm{Sym}^2(\mathbf{P}^1) \simeq \mathbf{P}^2$. Moreover,

$$\boxed{D_x = T_{g(x)}\Gamma.}$$

To see this geometrically, intersect D_x with Γ. A point of Γ has the form $g(y) = 2y$. It belongs to D_x exactly when the doubled divisor $2y$ contains x, which forces $y = x$. Hence D_x meets the conic only at $g(x)$. By Bézout, the intersection multiplicity there is two, so D_x is the tangent line to Γ at $g(x)$.

Remark 1. *There is also an elementary affine description. On* $\mathrm{Sym}^2(\mathbb{A})$ *use the symmetric coordinates*

$$S = y + z, \qquad P = yz.$$

Then

$$\Gamma : \quad P = \frac{S^2}{4},$$

and the condition that the quadratic divisor $y + z$ contain x is

$$P = xS - x^2.$$

Thus

$$D_x : \quad P = xS - x^2,$$

which is the tangent line to the parabola $P = S^2/4$ at

$$g(x) = (2x, x^2).$$

4 The second family of lines: a pencil through $g(\infty)$

For finite $x \in \mathbb{A}$, the zero-sum interpretation gives

$$E_x = \overline{\{y + z : y + z = -x\}}.$$

In the affine symmetric coordinates (S, P), this is simply

$$E_x : \quad S = -x.$$

Thus the finite members of the family are parallel affine lines. Their projective closures all pass through the same point at infinity, namely

$$p_\infty := g(\infty) = 2\infty \in \Gamma.$$

To find the other intersection of E_x with Γ, let $g(t) = 2t \in \Gamma$. Then

$$g(t) \in E_x \quad \Longleftrightarrow \quad x + 2t = 0.$$

Define the automorphism

$$\tau : \mathbf{P}^1 \longrightarrow \mathbf{P}^1$$

by

$$\boxed{x + 2\tau(x) = 0}$$

on $\mathbb{A}$, and by $\tau(\infty) = \infty$. In an affine coordinate with origin 0,

$$\tau(x) = -\frac{x}{2}.$$

The factor 2 appears because $g(t)$ represents the doubled divisor $t + t$.

It follows that

$$E_x \cap \Gamma = p_\infty + g(\tau(x)),$$

and hence

$$\boxed{E_x = \overline{p_\infty \, g(\tau(x))}.}$$

For $x = \infty$, the two points coalesce, so the limiting secant is the tangent line

$$E_\infty = T_{p_\infty}\Gamma.$$

Since also $D_\infty = T_{p_\infty}\Gamma$, one has

$$E_\infty = D_\infty.$$

Thus the two families are

$$\boxed{D_x = T_{g(x)}\Gamma, \qquad E_x = \overline{p_\infty \, g(\tau(x))}.}$$

The D_x are the tangent lines to Γ, while the E_x form the complete pencil of lines through p_∞.

5 Their intersections trace an auxiliary conic

Let

$$\alpha := \tau^{-1} : \mathbf{P}^1 \longrightarrow \mathbf{P}^1.$$

On the affine line $\mathbb{A}$, the point $\alpha(x)$ is characterized by

$$2x + \alpha(x) = 0;$$

equivalently, $\alpha(x) = -2x$. The automorphisms α and τ are inverse because $\tau(x)$ is characterized by

$$x + 2\tau(x) = 0.$$

Both fix ∞.

For finite x, let

$$h(x) := D_x \cap E_x.$$

Because $h(x) \in D_x$, the quadratic divisor $h(x)$ contains x, so it has the form

$$h(x) = x + y$$

for a unique $y \in \mathbf{P}^1$. Since $h(x) \in E_x$, one also has

$$x + h(x) = 2x + y \in H.$$

The zero-sum description of H therefore gives $2x + y = 0$, and hence

$$\boxed{h(x) = x + \alpha(x).}$$

This formula extends over $x = \infty$ and gives

$$h(\infty) = 2\infty = p_\infty.$$

Thus it supplies the canonical limiting intersection at the unique fiber where $D_\infty = E_\infty$.

Now consider the symmetrization morphism

$$q : \mathbf{P}^1 \times \mathbf{P}^1 \longrightarrow \mathrm{Sym}^2(\mathbf{P}^1) \simeq \mathbf{P}^2, \qquad q(a, b) = a + b,$$

and the graph

$$G_\alpha := \{(x, \alpha(x)) : x \in \mathbf{P}^1\} \subset \mathbf{P}^1 \times \mathbf{P}^1.$$

The preceding formula says that h is the restriction of q to G_α:

$$\mathbf{P}^1 \simeq G_\alpha \xrightarrow{q} \mathbf{P}^2.$$

The graph G_α has bidegree $(1, 1)$, and

$$q^*\mathcal{O}_{\mathbf{P}^2}(1) \simeq \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1, 1).$$

Consequently,

$$h^*\mathcal{O}_{\mathbf{P}^2}(1) \simeq \mathcal{O}_{\mathbf{P}^1}(1) \otimes \alpha^*\mathcal{O}_{\mathbf{P}^1}(1) \simeq \mathcal{O}_{\mathbf{P}^1}(2).$$

The map h is generically one-to-one. Indeed, an equality

$$x + \alpha(x) = x' + \alpha(x')$$

of unordered pairs gives, generically, either $x' = x$, or $x' = \alpha(x)$ and $\alpha(x') = x$. The second possibility would imply $\alpha^2(x) = x$. Since α^2 is not the identity, this occurs only at exceptional points. It follows that the image of h has degree two. Being an irreducible plane curve of degree two, it is a smooth conic; write

$$R := h(\mathbf{P}^1) \subset \mathbf{P}^2.$$

Moreover,

$$h : \mathbf{P}^1 \xrightarrow{\sim} R$$

is an isomorphism and $p_\infty = h(\infty)$ lies on R.

The two original line families, now viewed as chords of R

The chord description of D_x follows directly from the interpretation of points of $\mathrm{Sym}^2(\mathbf{P}^1)$ as degree-two divisors. Recall that

$$h(x) = x + \alpha(x), \qquad \alpha = \tau^{-1}.$$

The divisor $h(x)$ contains x, so $h(x) \in D_x$. On the other hand,

$$h(\tau(x)) = \tau(x) + \alpha(\tau(x)) = \tau(x) + x,$$

and this divisor also contains x. Hence $h(\tau(x)) \in D_x$ as well. If x is not fixed by τ, these are two distinct points of the conic R, and the unique line through them is therefore D_x:

$$\boxed{D_x = \overline{h(x)h(\tau(x))}.}$$

At a fixed point of τ, the two endpoints coalesce. Letting nearby points approach x, the chords $\overline{h(y)h(\tau(y))} = D_y$ converge to the tangent line to the smooth conic R at $h(x)$, while the family D_y converges to D_x. Thus the same formula holds everywhere, with a coincident chord interpreted as a tangent.

There is a parallel description of E_x. The point $h(x)$ lies on E_x by construction, while $p_\infty = h(\infty)$ lies on every E_x. Equivalently, for finite t,

$$h(t) \in E_x \quad \Longleftrightarrow \quad x + t + \alpha(t) = 0 \quad \Longleftrightarrow \quad t = x,$$

and the second projective intersection occurs at $t = \infty$. Therefore

$$E_x \cap R = h(x) + p_\infty,$$

and hence

$$\boxed{E_x = \overline{h(x)p_\infty}.}$$

At $x = \infty$ this is the tangent line $T_{p_\infty}R$.

Thus, relative to the auxiliary conic R,

$$\boxed{\begin{aligned} E_x &= \text{the chord from } h(x) \text{ to the fixed point } p_\infty, \\ D_x &= \text{the chord from } h(x) \text{ to the moving point } h(\tau(x)). \end{aligned}}$$

6 The residual-intersection map

Take a point $(x, Q) \in U$. Since

$$h(x) \in D_x \cap E_x$$

and both D and E have been removed, one has $Q \neq h(x)$. Thus the line

$$\ell_{x,Q} := \overline{h(x)Q}$$

is well defined.

Because $h(x) \in R$, the line $\ell_{x,Q}$ meets R in $h(x)$ and one residual point. Write

$$\ell_{x,Q} \cap R = h(x) + \rho(x, Q)$$

scheme-theoretically. If $\ell_{x,Q}$ is tangent to R at $h(x)$, then $\rho(x, Q) = h(x)$, counted with multiplicity.

This defines a morphism

$$\rho : U \longrightarrow R.$$

Geometrically, ρ projects Q to the conic R from the moving center $h(x)$ and takes the second intersection point.

Now

$$\rho(x, Q) = p_\infty$$

if and only if

$$\overline{h(x)Q} = \overline{h(x)p_\infty} = E_x,$$

which is equivalent to $Q \in E_x$. Since E has been removed,

$$\rho(U) \subset R \setminus \{p_\infty\}.$$

Projection from p_∞ identifies the punctured conic with an affine line:

$$L : R \setminus \{p_\infty\} \xrightarrow{\sim} \mathbf{A}^1.$$

Equivalently, a point $s \in R \setminus \{p_\infty\}$ is sent to the secant line $\overline{p_\infty s}$ in the pencil through p_∞; the omitted point p_∞ corresponds to the tangent direction $T_{p_\infty} R$.

We therefore obtain

$$\boxed{\pi := L \circ \rho : U \longrightarrow \mathbf{A}^1.}$$

Explicitly,

$$\boxed{\pi(x, Q) = L\Big(\text{the residual point of } \overline{h(x)Q} \cap R\Big).}$$

7 The fibers of π

Fix a point

$$s \in R \setminus \{p_\infty\}.$$

The closure of the fiber over s inside X is

$$M_s := \{(x, Q) : Q \in \overline{h(x)s}\}.$$

As x varies, the point $h(x)$ runs through R, and the lines $\overline{h(x)s}$ run through the complete pencil of lines through s. When $h(x) = s$, the corresponding limiting line is the tangent T_sR. Thus

$$M_s \simeq \mathrm{Bl}_s \, \mathbf{P}^2,$$

the incidence surface of points lying on lines through s. Equivalently, M_s is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1_x$, whose fiber over x is $\overline{h(x)s}$.

Inside M_s, the divisor E removes the section

$$\Sigma := \{(x, h(x)) : x \in \mathbf{P}^1\}.$$

Indeed, $E_x = \overline{h(x)p_\infty}$ meets $\overline{h(x)s}$ at $h(x)$ because $s \neq p_\infty$. Under the blow-down $M_s \to \mathbf{P}^2$, the section Σ is the proper transform of R.

The divisor D removes one entire ruling fiber. Namely, there is a unique $x_s \in \mathbf{P}^1$ such that

$$h(\tau(x_s)) = s.$$

At that value,

$$D_{x_s} = \overline{h(x_s)h(\tau(x_s))} = \overline{h(x_s)s},$$

so the full ruling over x_s lies in D. For every other x, the lines D_x and $\overline{h(x)s}$ meet only at $h(x)$, which is already contained in the removed section Σ.

Hence

$$\boxed{\pi^{-1}(s) = M_s \setminus (\Sigma \cup F_{x_s}),}$$

where F_{x_s} is one ruling fiber.

Removing F_{x_s} changes the base from $\mathbf{P}^1$ to

$$\mathbf{P}^1 \setminus \{x_s\} \simeq \mathbf{A}^1.$$

Removing Σ removes one point from every remaining $\mathbf{P}^1$-fiber, leaving an affine line. Thus

$$\pi^{-1}(s) \longrightarrow \mathbf{A}^1$$

is an $\mathbf{A}^1$-bundle. More precisely, it is the complement of a section in a $\mathbf{P}^1$-bundle over $\mathbf{A}^1$, hence a torsor under a line bundle. Every line bundle on $\mathbf{A}^1$ is trivial and $H^1(\mathbf{A}^1, \mathcal{O}_{\mathbf{A}^1}) = 0$, so the torsor is trivial. Therefore

$$\boxed{\pi^{-1}(s) \simeq \mathbf{A}^2.}$$

Equivalently,

$$\boxed{\mathrm{Bl}_s \, \mathbf{P}^2 \setminus (\widetilde{R} \cup \text{one ruling fiber}) \simeq \mathbf{A}^2.}$$

8 The global affine three-space structure

The same geometry describes the entire space U. Define

$$B := \Big\{(x, s) \in \mathbf{P}^1 \times (R \setminus \{p_\infty\}) : s \neq h(\tau(x))\Big\}.$$

The excluded condition $s = h(\tau(x))$ is exactly the condition that the line $\overline{h(x)s}$ coincide with D_x, in which case the entire line has been removed.

For fixed s, exactly one value of x is excluded. Thus

$$B \longrightarrow R \setminus \{p_\infty\} \simeq \mathbf{A}^1$$

is an $\mathbf{A}^1$-bundle. As above, it is trivial, so

$$B \simeq \mathbf{A}^2.$$

There is a natural morphism

$$U \longrightarrow B, \qquad (x, Q) \longmapsto (x, \rho(x, Q)).$$

For fixed $(x, s) \in B$, the point Q varies on

$$\overline{h(x)s} \setminus \{h(x)\} \simeq \mathbf{A}^1.$$

Thus $U \to B$ is again an $\mathbf{A}^1$-bundle arising as the complement of a section in a $\mathbf{P}^1$-bundle. Since $B \simeq \mathbf{A}^2$, its relevant line bundles and additive torsors are trivial. Consequently,

$$\boxed{U \simeq \mathbf{A}^3.}$$

Under this identification, π becomes the projection

$$\mathbf{A}^3 \longrightarrow \mathbf{A}^1$$

onto the residual-intersection parameter.

9 Summary of the construction

The construction can be compressed into the following sequence:

$$\boxed{\begin{array}{c} \Gamma = \{2x\} \subset \mathrm{Sym}^2(\mathbf{P}^1), \\[1ex] D_x = T_{g(x)}\Gamma, \qquad E_x = \overline{p_\infty\, g(\tau(x))}, \\[1ex] h(x) = D_x \cap E_x, \qquad R = \overline{\{h(x) : x \in \mathbf{P}^1\}}, \\[1ex] E_x = \overline{h(x)p_\infty}, \qquad D_x = \overline{h(x)h(\tau(x))}, \\[1ex] (x, Q) \longmapsto \text{the second point of } \overline{h(x)Q} \cap R, \\[1ex] R \setminus \{p_\infty\} \simeq \mathbf{A}^1, \\[1ex] \pi : U \to \mathbf{A}^1, \qquad \pi^{-1}(s) \simeq \mathbf{A}^2, \qquad U \simeq \mathbf{A}^3. \end{array}}$$

The central geometric idea is that the two boundary divisors determine two moving lines in every $\mathbf{P}^2$-fiber. Their intersections trace a second conic R. Projection from the moving point $h(x)$ to this conic produces a residual-intersection parameter. The divisor E removes the fixed residual value p_∞, while the divisor D removes one value of the original $\mathbf{P}^1$-parameter in every residual fiber. What remains is successively an affine line over an affine line over an affine line.

A The action of α on the auxiliary conic in affine coordinates

It is convenient to describe the action of $\alpha = \tau^{-1}$ on the auxiliary conic inside the affine chart

$$\mathrm{Sym}^2(\mathbb{A}) \simeq \mathbb{A}^2_{S,P}, \qquad S = y + z, \quad P = yz.$$

In these coordinates the Veronese conic is

$$\Gamma : \quad P = \frac{S^2}{4}.$$

The diagonal action of α on unordered pairs induces the affine map

$$\mathrm{Sym}^2(\alpha) : \mathbb{A}^2_{S,P} \longrightarrow \mathbb{A}^2_{S,P}, \qquad (S, P) \longmapsto (-2S, 4P).$$

Indeed, if $\alpha(t) = -2t$, then on a divisor $y + z$ one has

$$S = y + z \longmapsto -2(y + z) = -2S, \qquad P = yz \longmapsto (-2y)(-2z) = 4P.$$

This affine map preserves the conic R. More precisely, if

$$h : \mathbb{A} \longrightarrow R, \qquad h(x) = x + \alpha(x),$$

then

$$\mathrm{Sym}^2(\alpha)(h(x)) = \alpha(x) + \alpha^2(x) = h(\alpha(x)).$$

So the restriction of $\mathrm{Sym}^2(\alpha)$ to R is simply the conjugate of α by the parametrization h:

$$\mathrm{Sym}^2(\alpha)|_R = h \circ \alpha \circ h^{-1}.$$

In particular, if one uses the affine coordinate S on R, then the action is again multiplication by -2.